Exponents and bounds for uniform spanning trees in d dimensions

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Uniform spanning trees are a statistical model obtained by taking the set of all spanning trees on a given graph (such as a portion of a cubic lattice in d dimensions), with equal probability for each distinct tree. Some properties of such trees can be obtained in terms of the Laplacian matrix on the graph, by using Grassmann integrals. We use this to obtain exact exponents that bound those for the power-law decay of the probability that k distinct branches of the tree pass close to each of two distinct points, as the size of the lattice tends to infinity.

In graph theory, a tree on a graph is a connected subset of the vertices and edges without cycles, and a spanning tree is a tree that includes all n vertices of the graph (it must then have $n-1$ edges). Results for the number of spanning trees on a given graph go back to the nineteenth century (see e.g. Ref. [1]). If each spanning tree is given equal probability, we obtain uniform spanning trees. In this paper, we consider uniform spanning trees on (a portion of) the square, cubic, or hypercubic lattice in d dimensions. One would like to characterize the fractal properties of the trees as the size (number of vertices) of the lattice goes to infinity. One characteristic is the probability that two well-separated points are nearly connected by $k = 1$, 2, 3, ... distinct branches of the tree, or alternatively by distinct paths along the tree, and these are expected to behave as power laws that are described by critical exponents. We will study these by methods based on the classical results, and obtain some exact exponents, which serve as bounds for more general ones. (In two dimensions, the exact results have been known for some time [2–6].) The motivation to consider this problem came from its connection to some optimization problems, which are in turn connected with the ground states of classical systems with quenched disorder, such as Ising spin glasses. In the two-dimensional case, there is also a connection with loop models, the $Q \to 0$ Potts model, and Coulomb gases in conformal field theory [2–4].

First we note that the result [variously attributed either to Kirchhoff (1847), or to Sylvester (1857), Borchardt (1860), and Cayley (1856)] for the number $\mathcal{N}$ of spanning trees on a graph can be written in the following generalized form:

$$\mathcal{N} = \operatorname{cof} \Delta^{(x_1,y_1)} = (-1)^{x_1+y_1} \det \Delta^{(x_1,y_1)}, \qquad (1)$$

where we have recalled the definition of the cofactor. Here x_1, $y_1 = 1$, 2, ... label the vertices in the graph, the matrix Δ (the lattice Laplacian) is defined as

$$\Delta(x,y) = \begin{cases} \deg x & \text{if } x = y, \\ -t & \text{if } x \text{ and } y \text{ are connected by } t \text{ edges}, \\ 0 & \text{otherwise}, \end{cases} \qquad (2)$$

and $\Delta^{(x_1,y_1)}$ means the minor of (x_1,y_1), i.e. Δ with row x_1 and column y_1 deleted. For $x_1 = y_1$, this reduces to

the better known result. The effect of deleting a row and column is to remove the zero mode that would otherwise cause the determinant of Δ to vanish.

The result generalizes further to a relation that involves the number $\mathcal{N}^{(x_1 y_1, x_2 y_2, \ldots, x_k y_k)}$ of spanning subgraphs without circuits with k components, and with x_i, y_i in the same component for each i (we will assume that all x_i, y_i are distinct). The result is

$$\operatorname{cof} \Delta^{(x_1 \ldots x_k, y_1 \ldots y_k)} = \\ \pm \sum_{P \in S_k} \mathcal{N}^{(x_1 y_{P(1)}, x_2 y_{P(2)}, \ldots, x_k y_{P(k)})} \operatorname{sgn} P. \qquad (3)$$

Here the cofactor is again

$$(-1)^{\sum_{i=1}^{k}(x_i+y_i)} \det \Delta^{(x_1 \ldots x_k, y_1 \ldots y_k)}, \qquad (4)$$

where rows x_i and columns y_i have been deleted, and P runs over permutations of k symbols. The overall sign on the right hand side depends on the details of how the vertices are labelled and is uninteresting. Both the generalizations are mentioned by Ivashkevich [5] (see also Ref. [7]), but he omits the signs in the cofactors. The results can be proved by an extension of the proof given for example in Ref. [1].

In the following we will consider a graph that is a bounded portion Λ of the d-dimensional cubic lattice $\mathbf{Z}^d$ (with edges that connect only nearest neighbors at Euclidean distance 1 in lattice units). We will be interested in the following property of a spanning tree. We choose two vertices x, y, together with a neighborhood of each. We assume that the neighborhoods are chosen in such a way that the boundary passes through some vertices, but no edges of Λ cross the boundary; all edges are either inside or outside. We take k vertices x_i on the boundary of the neighborhood x, and k vertices y_i on the boundary of that of y. In practise, this can be satisfied using neighborhoods that are approximately balls of radius of order $k^{1/(d-1)}$. We can now look at the part of the tree lying outside the two neighborhoods; this amounts to a forest of trees, with each tree rooted on both the boundaries of the neighborhoods of x and y. We ask whether, for each i, the points x_i, y_i lie in the same connected component in this forest, and are in a distinct component from any other pair x_j, y_j. If so, then in terms of the original tree the x_is are connected to the corresponding
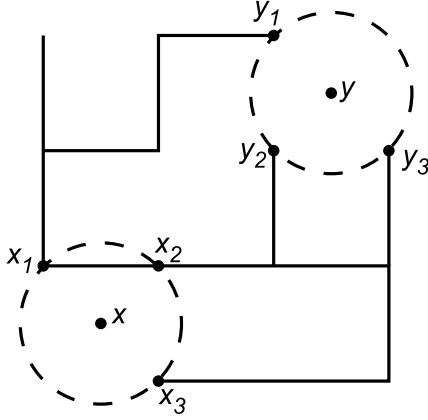
FIG. 1. Example of a spanning tree on a portion of the square lattice, with points x, y and a neighborhood of each marked, with vertices x_1, x_2, x_3 and y_1, y_2, y_3 on the boundary of each shown. In this example, there are paths, lying outside the neighborhoods, that connect x_i to y_i for each i, as required, and those paths are distinct, but there are only two distinct branches (connected components) outside the neighborhoods. The pairs x_2, y_2 and x_3, y_3 do not lie on distinct branches.

y_is by branches of the tree that are distinct outside the two neighborhoods selected, and we call this "crossing k times (between specified points in the neighborhoods of x and y) by distinct branches". See Fig. 1. For given Λ, neighborhoods of x and y, and points x_i, y_i, denote the number of such trees $\mathcal{N}_{\text{branches}}^{(x_1y_1, x_2y_2, \ldots, x_ky_k)}$. Then we also define corresponding probabilities,

$$\mathcal{P}_{\text{branches}}^{(x_1y_1, x_2y_2, \ldots, x_ky_k)} = \mathcal{N}_{\text{branches}}^{(x_1y_1, x_2y_2, \ldots, x_ky_k)}/\mathcal{N}. \quad (5)$$

It is obvious that there is a relation between this definition, and the k-component spanning subgraphs without circuits considered before, if the latter are defined on Λ^-, that is Λ with the interiors of the neighborhoods of x and y removed. Given a k-component spanning subgraph of Λ^-, a spanning tree of Λ can be obtained by adding $k-1$ edges, each inside either the neighborhood of x or of y, together with one edge for each vertex in $\Lambda - \Lambda^-$. If we also assume that there are exactly k vertices on the boundaries of each of the two neighborhoods, then $\mathcal{N}_{\text{branches}}^{(x_1y_1, x_2y_2, \ldots, x_ky_k)}$ is obtained by summing over all the ways to add edges to each k-component spanning subgraph of Λ^- (with x_i and y_i in the ith component for all i) to obtain a spanning tree of Λ whose branches are distinct outside the two neighborhoods, and then also summing over the k-component subgraphs used in this construction. As the maximum possible number of ways to add edges is limited by the size of the neighborhoods used, $\mathcal{N}_{\text{branches}}^{(x_1y_1, x_2y_2, \ldots, x_ky_k)}$ is only slightly larger than the number $\mathcal{N}^{(x_1y_1, x_2y_2, \ldots, x_ky_k)}$ of k-component spanning graphs of Λ^-. (More precisely, there is a bound $\mathcal{N}_{\text{branches}}^{(x_1y_1, x_2y_2, \ldots, x_ky_k)} \le c_{k,d} \mathcal{N}^{(x_1y_1, x_2y_2, \ldots, x_ky_k)}$ where $c_{k,d}$ is independent of the distance from x to y.)

Another possible definition of crossing from x to y would require only that the k crossings of the spanning tree be made by k *paths* on the tree that are distinct (have no edges in common) outside the neighborhoods (for $k = 1$ these are the same thing, and the probability is one). See Fig. 1. These numbers $\mathcal{N}_{\text{paths}}^{(x_1y_1, x_2y_2, \ldots, x_ky_k)}$ can be obtained by adding edges to the k-component spanning subgraph of Λ^- in arbitrary positions in Λ, and so $\mathcal{N}_{\text{branches}}^{(x_1y_1, x_2y_2, \ldots, x_ky_k)} \le \mathcal{N}_{\text{paths}}^{(x_1y_1, x_2y_2, \ldots, x_ky_k)}$, and

$$\mathcal{N}^{(x_1y_1, x_2y_2, \ldots, x_ky_k)}/\mathcal{N} \le \mathcal{P}_{\text{branches}}^{(x_1y_1, x_2y_2, \ldots, x_ky_k)}$$
$$\le \mathcal{P}_{\text{paths}}^{(x_1y_1, x_2y_2, \ldots, x_ky_k)}. \quad (6)$$

These continue to hold even if there are more than k vertices on the boundary of each neighborhood.

We will be interested in the scaling limit in which we first let $\Lambda \to \mathbf{Z}^d$ (i.e. the system size tends to infinity) with x, y fixed, then let the Euclidean distance $|x - y|$ (in lattice units) become large. In this limit, the leading behavior of any of the ratios or probabilities just defined will behave as a power law, $\mathcal{P} \propto |x-y|^{-2X_k}$, where X_k is a scaling dimension (which also depends on d). However, in the antisymmetrized combination that appears in eq. (3), the leading part of

$$\mathcal{N}^{(x_1y_1, x_2y_2, \ldots, x_ky_k)}/\mathcal{N} \propto |x-y|^{-2X_k^{k\,\text{components}}} \quad (7)$$

may be cancelled, so that a subleading power (with a larger exponent X_k^{antisymm}) may be dominant. Thus in general the exponents must obey the inequalities

$$X_k^{\text{antisymm}} \ge X_k^{k\,\text{components}} \ge X_k^{\text{branches}} \ge X_k^{\text{paths}}. \quad (8)$$

As the addition of edges in the neighborhood of x or y contributes only a constant factor to the numbers $\mathcal{N}_{\text{branches}}$, we have that $X_k^{k\,\text{components}} = X_k^{\text{branches}}$. For the two-dimensional case, the tree branches must enter the neighborhood of x in a sequence, and similarly at y. As the tree branches that cross from x to y cannot intersect because Λ is planar, the only nonzero terms in eq. (3) are for Ps that differ only by a cyclic permutation. For k odd, all cyclic permutations of k objects are even permutations, so all terms have the same sign, as if the sum were symmetrized. Therefore in $d = 2$, $X_k^{\text{antisymm}} = X_k^{k\,\text{components}}$ for k odd. We also note that in $d = 2$ dimensions, the boundary of a "thickened" tree is a nonintersecting dense loop, and the crossing by distinct branches corresponds to crossing by the loop, $2k$ times.

There is one further subtlety that must be mentioned. We have defined ratios and probabilities on finite graphs, followed by an infinite-volume limit. Each finite graph is spanned by a single tree, by construction, but as the volume increases, a path from one vertex to another in a typical tree may involve larger and larger excursions, so that for infinite volume, two vertices might be connected

only "at infinity", and then they can be regarded as not connected. Then the limiting measure is for a forest of trees, not a single tree. It turns out that for $d \leq 4$, there is a single tree in the infinite-volume limit, while for $d > 4$ there are infinitely many (infinite-size) trees [8] (these statements hold with probability one). Crossing probabilities by k distinct branches or k distinct paths can be defined here as well, with crossing not allowed to be "through infinity"; we denote exponents under this condition by "$< \infty$". These probabilities will be less than or equal to the corresponding ones defined above, and so the related exponents obey $X_k^{\mathrm{branches}<\infty} \geq X_k^{\mathrm{branches}}$, $X_k^{\mathrm{paths}<\infty} \geq X_k^{\mathrm{paths}}$. One has $X_1^{<\infty} = (d-4)/2$ for $d > 4$ (Ref. [8], Thm. 4.2), which because of the way this probability is normalized (any vertex is on some tree) is consistent with the belief that the Hausdorff dimension of any of the trees in the forest is 4 for $d > 4$. It would be interesting to obtain the remaining exponents $X_k^{\mathrm{paths}<\infty}$ and $X_k^{\mathrm{branches}<\infty}$ for $k > 1$ also. In any case, the distinction in definition disappears for $d \leq 4$ where there is a single spanning tree.

We now turn to the calculation of the antisymmetrized combinations of $\mathcal{N}$s. According to eq. (3), the antisymmetrized sum of the relevant ratios is given by a ratio of cofactors:

$$\frac{\mathrm{cof}\, \Delta^{(x_1 \ldots x_k, y_1 \ldots y_k)}}{\mathrm{cof}\, \Delta^{(x_1, y_1)}}$$
$$= \pm \sum_{P \in S_k} \mathcal{N}^{(x_1 y_{P(1)}, x_2 y_{P(2)}, \ldots, x_k y_{P(k)})} \mathrm{sgn}\, P / \mathcal{N}. \quad (9)$$

As is well-known, such cofactors can be obtained from Gaussian integrals over Grassmann variables ψ, ψ^*,

$$\mathrm{cof}\, \Delta^{(x_1 \ldots x_k, y_1 \ldots y_k)}$$
$$= \pm \int \prod_x d\psi_x d\psi_x^* \, \psi_{x_1} \cdots \psi_{y_k}^* e^{\sum_{x,y} \psi_x \Delta(x,y) \psi_y^*}, \quad (10)$$

where the overall sign is determined by the order of the Grassmann integrations (not by the selected values of x_i, y_i). In the limits $\Lambda \to \mathbf{Z}^d$, followed by the limit of x and y far apart, the cofactors above become Gaussian integrals for a continuum massless complex scalar Fermi field. The equation of motion for the Fermi field ψ is simply $\Delta \Psi = 0$, where $\Delta = \sum_{\mu=1}^d \partial_\mu \partial_\mu$ is the Laplacian in d dimensions. As all $x_i \to x$, the ratio of cofactors becomes a sum of correlation functions of operators of the form

$$\mathcal{O}(x) = \psi \partial_\mu \psi \cdots \partial_{\mu_1 \mu_2 \ldots} \psi \quad (11)$$

(with k ψs, and where $\partial_{\mu_1 \mu_2 \ldots} = \partial_{\mu_1} \partial_{\mu_2} \cdots$) at x, with a similar operator at y with ψ^* in place of ψ. The undifferentiated ψ and ψ^* are necessary to cancel the zero mode, both in the numerator and denominator of the correlation function. The remaining integrals can be simply expressed (using Wick's theorem) in terms of sums of products of derivatives of Green's functions $G(x,y) = \Delta^{-1}(x,y)$ for the scalar field ψ in d dimensions, and the required scaling limit of this expression exists; one has $G(x,y) \propto |x-y|^{d-2}$ for $d > 2$. Thus the scaling dimension of ψ or ψ^* is $(d-2)/2$ in d dimensions, and an operator of the above form $\mathcal{O}$ has scaling dimension $X_k^{\mathrm{antisymm}} = \dim \mathcal{O}$ equal to $(k-1)(d-2)/2$ plus the number of partial derivatives in $\mathcal{O}$ (note that we replaced k by $k-1$ because the subtracted zero mode does not contribute to scaling). This implies that for $k = 1$, the operator has dimension zero, which is correct as a spanning tree connects any two points x, y. It will be convenient to define $\dim' \mathcal{O} = \dim \mathcal{O} - (k-1)(d-2)/2$.

To find the operator that contributes the leading behavior of the correlation function for a given k, we must use as few derivatives as possible. Further, the equation of motion implies that any trace such as $\sum_\mu \partial_{\mu\mu\mu_3 \ldots} \psi$ vanishes. Then the leading term $\mathcal{O}$ is a product in which each multi-index partial derivative $\partial_{\mu_1 \mu_2 \ldots} \psi$ is a traceless symmetric tensor, and the total degree (number of derivatives) in the product is as low as possible. Because of the anticommutation of the ψs, $\mathcal{O}$ vanishes unless the traceless symmetric tensors $\partial_{\mu_1 \mu_2 \ldots} \psi$ are linearly independent. We notice that the traceless symmetric tensors of given rank (degree) in dimension d form an irreducible representation of the rotation group in d dimensions, $SO(d)$. In general, we expect that the leading part of the crossing probability $\mathcal{P}_k$ transforms as a scalar under rotations. A scalar (rotationally-invariant) operator can be obtained if the product includes either all or none of the members of a complete linearly-independent set of traceless symmetric tensors for each degree (rank) l. Making the minimal choices of the degrees, we notice that this is analogous to filling states for fermions, where the single fermion states correspond to the symmetric traceless tensors. These tensors transform the same way as "hyperspherical harmonics", which span the space of functions on a sphere S^{d-1}. This can be seen easily by representing each ∂_μ by a component of a vector x_μ, which transforms the same way, and the traces can be excluded if we assume that $\sum_\mu x_\mu x_\mu$ is constant, so that tensors with non-vanishing trace are equivalent to lower-degree functions. Then the symmetric functions in x_μ under this condition are simply the functions on S^{d-1}. If each fermion on S^{d-1} has a kinetic energy that is equal to the angular momentum l, then the many-fermion state with lowest total kinetic energy for given number of fermions k corresponds to using the lowest total degree. The case where all traceless symmetric tensors of each (lowest) degree are used corresponds to filling a Fermi sea by filling the lowest shells up to angular momentum (degree of the traceless symmetric tensor) L. The total kinetic energy corresponds to the scaling dimension $\dim' \mathcal{O}$. When the lowest states are all filled, but the topmost shell is only partially filled, the scaling dimension interpolates linearly between the values it takes for filled shells. We point out

that the use of fermions on the sphere is more than an analogy, as the field theory of fermions on S^{d-1} with time t corresponds to the original problem in radial quantization, obtained by conformally mapping $\mathbf{R}^d$ to $S^{d-1} \times \mathbf{R}$ by a logarithmic change of variable, so that the dilatation operator becomes the Hamiltonian for the radial evolution.

We define $N(l, d)$ to be the dimension of the space of traceless symmetric tensors of degree l in dimension d. When the shells are filled up to degree L, the preceding considerations lead immediately to the relations

$$k = \sum_{l=0}^{L} N(l, d), \qquad (12)$$

$$\dim' \mathcal{O}_L = \sum_{l=0}^{L} l N(l, d). \qquad (13)$$

$N(l, d)$ is given for all $l \geq 0$ by

$$N(l, d) = \binom{l + d - 1}{l} - \binom{l + d - 3}{l - 2}. \qquad (14)$$

Here the first term is the number of symmetric tensors, and the subtraction is for removing the traces. From the binomial coefficients one sees that $N(l, d)$ is a polynomial in l of degree $d-2$ for $d \geq 2$, and hence k is a polynomial in L of degree $d-1$, and $\dim' \mathcal{O}_L$ is a polynomial of degree d.

The leading behavior for l large is

$$N(l, d) \sim \frac{2l^{d-2}}{(d-2)!} \qquad (15)$$

(throughout this paper, we use notation $X \sim Y$ as $Z \to \infty$ in the strict sense: $\lim_{Z\to\infty} X/Y = 1$). Then we find

$$k \sim \frac{2L^{d-1}}{(d-1)!}, \qquad (16)$$

$$\dim' \mathcal{O}_L \sim \frac{2L^d}{d(d-2)!}, \qquad (17)$$

and hence

$$\dim' \mathcal{O}_L \sim \dim \mathcal{O}_L \sim \frac{d-1}{d} \left[\frac{(d-1)!}{2} \right]^{\frac{1}{d-1}} k^{\frac{d}{d-1}}, \quad (18)$$

for the values of k specified. As mentioned above, for other values of k, the scaling dimension (now for an operator with nonzero spin in general) lies on a piecewise linear continuous curve that interpolates the values above, and lies above that given implicitly by eqs. (12), (13) as polynomials in L, which are trivially extended to continuous values.

By the inequalities (8), this result gives only an upper bound on the leading exponents X_k^{branches} or X_k^{paths}. However, the general formulas do give the exact exponents X_k^{antisymm} for some, possibly subleading, terms in

the probability. The rate of growth of the dimensions X_k^{paths} on the tree to cross from x to y was shown rigorously to be less than of order $k^{d/(d-1)}$ in Ref. [9]. We obtain a bound with the same power, but now with a precise coefficient, and with subleading corrections. Note that the piecewise-linear curve for X_k^{antisymm} is very close to its lower envelope, close enough that they have the same average rate of growth, eq. (18).

We now consider the exact form of the dimensions obtained here for the k values given by eq. (12) in small d. In two dimensions, $N(l, 2) = 2$ ($l > 0$), and then

$$\dim \mathcal{O}_k = (k^2 - 1)/4 \qquad (19)$$

for k odd. This is in agreement with earlier results [2,3,5,6]. (Note that the scaling exponent for the path connecting x to y along the tree [4] corresponds to the case $k = 2$, by considering the dual tree.) As emphasized above, for $d = 2$ and k odd, the arguments in this paper produce the *exact* $X_k^{\text{antisymm}} = X_k^{\text{branches}} = (k^2 - 1)/4$, not only a bound. Note that after replacing k by $k/2$, this result is the same as the "k-leg" dimension for k crossings by a dense polymer [10].

For $d = 3$, we have the familiar formula $N(l, 3) = 2l+1$, and then $k = (L + 1)^2$, so $L = \sqrt{k} - 1$. For the scaling dimensions, $\dim' \mathcal{O}_L = L(L + 1)(4L + 5)/6$, and

$$X_k^{\text{antisymm}} = \dim \mathcal{O}_k = \frac{2}{3}k^{3/2} - \frac{1}{6}k^{1/2} - \frac{1}{2}. \qquad (20)$$

For $d > 3$, one can similarly solve explicitly, but the results are not as simple (in particular, they are not polynomials in $k^{\frac{1}{d-1}}$). I am grateful to C. Tanguy for pointing out that eqs. (12), (13) can be summed in closed form for all d, and that for d odd, k is a polynomial in $[L + (d - 1)/2]^2$ of degree $(d - 1)/2$. Hence in the cases $d = 4$, 5, 7, and 9, $\dim \mathcal{O}_k$ can be expressed in terms of radicals in k. For $d = 6$, 8 and all $d \geq 10$, one meets polynomials of degree greater than four, and the results presumably cannot be expressed in terms of radicals.

It is tempting to believe that the results obtained here for the "filled shell" values of k, and their smooth (polynomial in L) extension to general k, might be the exact values of X_k^{branches} and X_k^{paths} for general dimension $d > 2$, as well as for $d = 2$. While it appears quite possible that $X_k^{\text{branches}} = X_k^{\text{paths}}$ in general, it is not at all clear that they equal X_k^{antisymm}, especially as the equality that holds in two dimensions could be obtained from a simple argument that all cyclic permutations of an odd number of objects are even, an argument that definitely does not go through in $d > 2$.

In conclusion, we have obtained, essentially rigorously, a precise upper bound on the exponents for k crossings of the uniform spanning tree on a finite graph in d dimensions, as well as some exact scaling dimensions in each dimension. However, we have not addressed the exponents in the uniform spanning forest which obtains on an infinite graph for $d > 4$.

[1] J.H. van Lint and R.M. Wilson, *A Course in Combinatorics* (Cambridge University, Cambridge, 1992), Chapter 34.

[2] B. Duplantier, J. Stat. Phys. **49**, 411 (1987); B. Duplantier and F. David, J. Stat. Phys. **51**, 327 (1988).

[3] H. Saleur, Nucl. Phys. B **382**, 486 (1992).

[4] S.N. Majumdar, Phys. Rev. Lett. **68**, 2329 (1992).

[5] E.V. Ivashkevich, J. Phys. A **32**, 1691 (1999).

[6] R. Kenyon, J. Math. Phys. **41**, 1338 (2000).

[7] V.B. Priezzhev, Sov. Phys. Usp. **28**, 1125 (1985).

[8] R. Pemantle, Ann. Prob. **19**, 1559 (1991).

[9] M. Aizenman, A. Burchard, C. M. Newman, and D. B. Wilson, Random Struct. Algorithms. **15**, 319 (1999).

[10] B. Duplantier and H. Saleur, Nucl. Phys. B **290**, 291 (1987).